

STRANGE BILLIARD TABLES

BY

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ABSTRACT. A billiard table is any compact convex body T in the plane bounded by a continuously differentiable curve ∂T . An idealized billiard ball is a point which moves at unit speed in a straight line except when it hits the boundary ∂T where it rebounds making the angle of incidence equal to the angle of reflection. A rather surprising phenomenon can happen on such a table.

1. Introduction. The idealized convex billiard table on which an idealized perfectly elastic billiard ball moves with unit speed is a well-known dynamical system [1]–[4]. The table T is any compact convex body in the plane bounded by a continuously differentiable curve ∂T which of course has a unique tangent line at each point. The ball is a point which moves at unit speed in a straight line except when it hits the boundary ∂T where it rebounds making the angle of incidence equal to the angle of reflection. To be precise, a trajectory of the billiard ball is any unit speed rectifiable curve $\alpha(t)$ such that if $\alpha(t_0) \notin \partial T$ then $\alpha(t)$ is part of a straight line for t near t_0 , and if $\alpha(t_0) \in \partial T$ then the left- and right-sided derivatives of α exist at t_0 and make equal and opposite angles with the tangent to ∂T at $\alpha(t_0)$. It has been observed in [2] that this defines a flow F_t on the space of “states of motion”. That is, for each initial position $p \in \text{interior } T = \text{int } T$ and direction of motion v ($v \in S = \text{set of unit vectors}$), $F_t(p, v)$ gives the position and forward direction of motion of the ball at time t .

We will show that F_t is not always well defined, and we will construct a billiard table for which it is possible to start a ball in the interior of the table and find it later outside of the table. Our example of a table with F_t not well defined is such that the boundary curve has three derivatives and nowhere vanishing curvature. In this example the third derivative exists everywhere, but it is not bounded. We show that if the boundary curve has nowhere vanishing curvature and a bounded third derivative, then F_t is well defined. We also show that, for all tables T , $F_t(p, v)$ is well defined for almost all $(p, v) \in \text{int } T \times S$.

Received by the editors December 12, 1975 and, in revised form, March 19, 1976.

AMS (MOS) subject classifications (1970). Primary 52A10.

Key words and phrases. Billiard table, plane convex body, dynamical system.

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The author is indebted to I. Kupka for many stimulating conversations.

2. F_t not necessarily well defined. To say that $F_{t_0}(p, \nu)$ is well defined is to assert that there is exactly one trajectory $\alpha(t)$ defined for $t \in [0, t_0]$ such that $\alpha(0) = p$ and $\alpha'(0) = \nu$. It appears that such a unique trajectory always exists for $p \in \text{interior } T$ and that it may be constructed step by step as follows. Travel at unit speed in the direction ν starting at p until you hit ∂T . Determine the new direction from the equal angles law and proceed at unit speed along a straight line in that direction until you hit ∂T again, and so on. The fallacy in this reasoning is that there is no guarantee that you will use up the time t_0 in this manner. This resembles Zeno's paradox. We will construct an example which shows that this possibility actually occurs, i.e., the ball can hit the boundary an infinite number of times in a finite amount of time. We will see that at the end of such an infinite sequence of hits the ball is approaching ∂T with zero angle. This leads to at least three ways to continue the trajectory. First, if the table is constructed appropriately the ball could follow a similar infinite hits backwards into the interior of T . Second, the ball could follow ∂T around at unit speed. Third, the ball may proceed at unit speed out along the tangent line. Each of these possibilities satisfies our definition of trajectory and hence it is possible for $F_t(p, \nu)$ not to be well defined because the desired trajectory α is not unique. This also explains how a ball starting in the interior of T might end up outside of T . The third possibility of continuing tangentially may appear unnatural but the point is that it does satisfy the given definition of trajectory, and this definition looked unobjectionable at first.

3. An example. The construction of the curve $\gamma = \partial T$ such that some trajectory will hit γ an infinite number of times in a finite period of time is quite transparent. We first choose the points of contact p_n of the special trajectory α and the boundary curve γ . We choose them on the unit circle with polar angles $\theta_n = n^{-1/2}$. This determines the polygonal trajectory α and the angle of reflection equals angle of incidence law gives the tangent line to γ at p_n . So we construct a small portion of γ through each point p_n so as to have the correct tangent at p_n . We choose these pieces γ_n to be portions of different unit circles (with appropriate centers) which go through the p_n . Next we connect up the γ_n by expressing them via polar equations $\gamma_n: r = \bar{\gamma}_n(\theta)$ and writing $\gamma: r = \bar{\gamma}(\theta)$ as a smooth average of γ_n and γ_{n+1} for $\theta_{n+1} \leq \theta \leq \theta_n$.

$$(1) \quad \bar{\gamma}(\theta) = \bar{\gamma}_n(\theta) + \alpha_n(\theta)(\bar{\gamma}_{n+1}(\theta) - \bar{\gamma}_n(\theta)).$$

The averaging functions α_n are constructed by

$$(2) \quad \alpha_n(\theta) = \alpha\left(\frac{\theta - \theta_{n+1}}{\theta_n - \theta_{n+1}}\right)$$

where $\alpha: [0, 1] \rightarrow [0, 1]$ is any fixed (independent of n) infinitely differentiable function such that $\alpha(t) = 1$ for $t \leq 1/3$ and $\alpha(t) = 0$ for $t \geq 2/3$. Actually we will use the above construction for $\bar{\gamma}$ only for $0 < \theta \leq \theta_{n_0}$ where n_0 is picked later after making some estimates. Furthermore we redefine $\bar{\gamma}_{n_0}$ by setting $\bar{\gamma}_{n_0}(\theta) = 1$. The other $\bar{\gamma}_n$'s, $n > n_0$, are as before. This allows us to complete the definition of $\bar{\gamma}$ by setting $\bar{\gamma}(\theta) = 1$ for $\theta \in [-\pi, \pi) - (0, \theta_{n_0})$.

It remains to be shown that for n_0 sufficiently large, $\bar{\gamma}$ has three derivatives and nowhere vanishing curvature. To this end we make the following estimates. Taylor expansions immediately give

$$(3) \quad \delta_n \equiv \theta_n - \theta_{n-1} = a_n n^{-3/2}, \quad \omega_n \equiv \delta_{n+1} - \delta_n = b_n n^{-5/2},$$

with $a_n \rightarrow -1/2$ and $b_n \rightarrow 3/4$ as $n \rightarrow \infty$. Let c be a bound for $|\alpha'(t)|$ and $|\alpha''(t)|$, $t \in [0, 1]$. Then from (2) and (3)

$$(4) \quad \begin{aligned} |\alpha'_n(\theta)| &\leq c/|\delta_n| \leq 3cn^{3/2}, \\ |\alpha''_n(\theta)| &\leq c/\delta_n^2 \leq 5cn^3, \end{aligned}$$

for $\theta \in [\theta_{n+1}, \theta_n]$ and n sufficiently large.

To obtain good estimates on the $\bar{\gamma}_n$ and their derivatives we note that the angle between S (S = the unit circle centered at 0) and γ_n at p_n is $\omega_n/4$. Also we can write

$$(5) \quad \gamma_n(\theta) = g(\theta - \theta_n, \omega_n)$$

where $r = g(\theta, \omega)$ is the polar equation for the unit circle passing through the point $(r, \theta) = (1, 0)$ and making an angle of $\omega/4$ with S at $(1, 0)$. It is easy to verify that g is well defined and infinitely differentiable on the compact set $D = \{(\theta, \omega) | -\pi/2 \leq \theta \leq \pi/2, -\pi/4 \leq \omega \leq \pi/4\}$. It is also clear that $g(0, \omega) = 1 = g(\theta, 0)$, for $(\theta, \omega) \in D$. If e is a positive upper bound for the absolute value of both partial derivatives g_{12} and g_{112} over D then the mean value theorem gives

$$(6) \quad \begin{aligned} |g_1(\theta, \omega)| &\leq e|\omega|, & |g_{11}(\theta, \omega)| &\leq e|\omega|, \\ |g(\theta, \omega) - 1| &\leq |g_1(\theta^*, \omega)\theta| \leq e|\omega||\theta|, \end{aligned}$$

where θ^* is between 0 and θ . Consequently from (5), (6) and (3)

$$(7) \quad \begin{aligned} |\bar{\gamma}'_n(\theta)| &\leq e|\omega_n| \leq e|b_n|n^{-5/2} \leq en^{-5/2}, \\ |\bar{\gamma}''_n(\theta)| &\leq e|\omega_n| \leq e|b_n|n^{-5/2} \leq en^{-5/2}, \\ |\bar{\gamma}_n(\theta) - 1| &\leq e|\omega_n||\theta - \theta_n| \leq e|\omega_n||\delta_n| \leq |ea_n b_n n^{-4}| \leq en^{-4}, \end{aligned}$$

for $\theta \in [\theta_{n+1}, \theta_{n-1}]$ and n sufficiently large.

Now if we pick n_0 sufficiently large and redefine $\bar{\gamma}_{n_0}$ to be given by $\bar{\gamma}_{n_0}(\theta) = 1$, then from (1), (4) and (7) we easily calculate that

$$\begin{aligned}
 |\bar{\gamma}(\theta) - 1| &\leq 3en^{-4}, \\
 |\bar{\gamma}'(\theta)| &\leq en^{-5/2} + 2en^{-5/2} + 3cn^{3/2}2en^{-4} \leq (3e + 6ce)n^{-5/2}, \\
 (8) \quad |\bar{\gamma}''(\theta)| &\leq en^{-5/2} + 2en^{-5/2} + 2 \cdot 3cn^{3/2}2en^{-5/2} + 5cn^32en^{-4} \\
 &\leq 2 \cdot 3cen^{-1}, \\
 \left| \frac{\bar{\gamma}''(\theta)}{\theta} \right| &\leq \frac{2 \cdot 3cen^{-1}}{(n+1)^{-1/2}},
 \end{aligned}$$

for $n \geq n_0$ and $\theta \in [\theta_{n+1}, \theta_n]$. It follows that $\lim_{\theta \rightarrow 0} \bar{\gamma}(\theta) = 1$, $\lim_{\theta \rightarrow 0} \bar{\gamma}'(\theta) = 0$, $\lim_{\theta \rightarrow 0} \bar{\gamma}''(\theta) = 0$. Hence $\bar{\gamma}'(0)$ and $\bar{\gamma}''(0)$ exist and $\bar{\gamma}$, $\bar{\gamma}'$ and $\bar{\gamma}''$ are continuous. Also $(\bar{\gamma}''(\theta) - \bar{\gamma}''(0))/(\theta - 0) = \bar{\gamma}''(\theta)/\theta \rightarrow 0$ as $\theta \rightarrow 0$. Hence $\bar{\gamma}'''(0)$ exists. We also see from (8) that if n_0 is picked sufficiently large then the curvature of γ can be made arbitrarily close to the curvature of S and hence never vanishes. This completes the construction of γ .

4. Sufficient conditions for F_t to be well defined. Consider the discrete analogue G of F_t defined as follows. Given a point $p \in \partial T$ and an angle θ , $0 < \theta < \pi$, set $G(p, \theta) = (p', \theta')$ where p' is the other intersection of the directed line L which passes through p making an angle θ with the tangent to ∂T at p (∂T is oriented counterclockwise) and θ' is the positive angle between L and the tangent to ∂T at p' . That is, if the billiard ball leaves from p making an angle θ with ∂T then p' is the next point of contact with the boundary ∂T and θ' is the next angle of reflection. Since T is convex, $G: \partial T \times (0, \pi) \rightarrow \partial T \times (0, \pi)$ is well defined. Set $G^n = G \circ G \circ \dots \circ G$ n times. Let $P_1: \partial T \times (0, \pi) \rightarrow \partial T$ and $P_2: \partial T \times (0, \pi) \rightarrow (0, \pi)$ be the natural projections, $P_1(p, \theta) = p$ and $P_2(p, \theta) = \theta$. Denote the Euclidean norm on the plane \mathbb{R}^2 by $\|\cdot\|$.

THEOREM 1. *If T is any billiard table and $\lim_{n \rightarrow \infty} P_1 \circ G^n(p, \theta) = q$ then*

- (a) $\lim_{n \rightarrow \infty} P_2 \circ G^n(p, \theta) = 0$ or π .
- (b) Either $\sum_{n=1}^{\infty} P_2 \circ G^n(p, \theta) < \infty$ or $\sum_{n=1}^{\infty} \pi - P_2 \circ G^n(p, \theta) < \infty$.
- (c) $\sum_{n=0}^{\infty} \|P_1 G^{n+1}(p, \theta) - P_1 G^n(p, \theta)\| < \infty$.

PROOF. Set up a Cartesian coordinate system with the origin at q , the positive x -axis in the direction of the tangent to γ at q and the positive y -axis in the direction of the principal normal of γ at q . We may express the curve γ locally about q as $y = g(x)$ for $|x| < \varepsilon$ where $\varepsilon > 0$ and g is continuously differentiable and satisfies $g(0) = 0$, $g'(0) = 0$. Call this portion of γ , $\bar{\gamma}$. By

picking ε smaller if necessary we may further require that $|g'(x)| \leq 10^{-6}$ for $|x| < \varepsilon$. Since $\lim_{n \rightarrow \infty} P_1 \circ G^n(p, \theta) = q$ there is an n_0 such that $q_n = P_1 \circ G^n(p, \theta) \in \bar{\gamma}$ for $n \geq n_0$. By replacing (p, θ) by $G^{n_0}(p, \theta)$ we may assume $n_0 = 0$. Set $\theta_n = P_2 \circ G^n(p, \theta)$ and x_n and y_n the x and y coordinates of q_n . It follows from the mean value theorem that the slope s of the line $\overline{q_n q_{n+1}}$ must satisfy $|s| \leq 10^{-6}$. We may now show that the sequence x_n is monotone. For if the x_n 's should change direction $x_n < x_{n+1} > x_{n+2}$ or $x_n > x_{n+1} < x_{n+2}$ for some n , then clearly the tangent to γ at q_{n+1} must be nearly vertical. But this is ruled out because $|g'(x)| \leq 10^{-6}$ for all x , $|x| < \varepsilon$. Hence the sequence x_n is monotone. By reversing the direction of the x -axis if necessary we may assume that x_n is monotone increasing. Since $\lim_{n \rightarrow \infty} x_n = 0$ we must have $x_n \leq 0$. Since the direction of motion monotonically changes by $2\theta_i$ at each point of contact, $\sum_{i=1}^{\infty} 2\theta_i$ is equal to the angle between the initial direction of motion and the tangent at q . Conclusions (a), (b), and $\sum_{n=1}^{\infty} \|q_{n+1} - q_n\| \leq$ arclength of γ from q_1 to q , and (c) now follows easily.

COROLLARY 2. *Given $(p, \theta) \in \partial T \times (0, \pi)$ and p_0 strictly between p and $q = P_1 G(p, \theta)$ on the straight line segment from p to q , set $v = (q - p)/\|q - p\|$. The following are equivalent.*

- (a) $F_t(p_0, v)$ is well defined for all $t \geq 0$.
- (b) $\sum_{n=0}^{\infty} \|P_1 G^{n+1}(p, \theta) - P_1 G^n(p, \theta)\| = \infty$.
- (c) $P_1 G^n(p, \theta)$ diverges.

PROOF. First we show not (c) \Rightarrow not (a). Assume $P_1 G^n(p, \theta)$ converges. By Theorem 1,

$$\sum_{n=0}^{\infty} \|P_1 G^{n+1}(p, \theta) - P_1 G^n(p, \theta)\| < \infty$$

and $P_2 G^n(p, \theta) \rightarrow 0$ or π . Hence starting at (p, θ) the ball is approaching a point $q \in \partial T$ tangentially at a finite time t_0 . Since we may continue the trajectory in at least two ways, around ∂T or out along the tangent line, $F_t(p_0, v)$ is not well defined for $t > t_0$.

Second we note that not (b) \Rightarrow not (c) follows immediately from Cauchy's convergence criteria.

Third we show that (b) \Rightarrow (a). Assuming (b) the naive construction and uniqueness proof discussed in §2 goes through and $F_t(p_0, v)$ is well defined for all $t \geq 0$.

The equivalence of (a), (b) and (c) follows. Q.E.D.

We will need some notation for Theorem 3 and its proof. The curvature of $\gamma = \partial T$ at the value of arclength s is denoted by $K(s)$. The principal normal to γ at s is denoted by $n(s)$. For any function of the arclength $r(s)$ we write $\dot{r} \equiv dr/ds$. We consider \mathbb{R}^2 imbedded as $\mathbb{R}^2 \times 0$ in \mathbb{R}^3 and then by identifying

the third axis with \mathbf{R} we can consider the vector cross product, denoted by $[\cdot, \cdot]$, as a function from $\mathbf{R}^2 \times \mathbf{R}^2$ into \mathbf{R} . We take the curve γ to be a periodic function $\gamma: \mathbf{R} \rightarrow \mathbf{R}^2$ with period = length γ . We will use $o(t-s)^2$ to signify any function $f(s, t)$ such that there exists a constant L such that $|f(s, t)| \leq L(t-s)^2$ for all $s, t \in \mathbf{R}$.

THEOREM 3. *If ∂T has a bounded third derivative and nowhere vanishing curvature, then $F_i(p, v)$ is well defined for all $(p, v) \in \text{int } T \times S$.*

PROOF. Assume the hypothesis and the contrary to the conclusion. It follows that $\dot{K}(s)$ exists and is bounded, where $K(s)$ is the curvature at s . By Corollary 2 and Theorem 1 there is a $p \in \partial T$ and θ , $0 < \theta < \pi$, such that

$$(9) \quad P_1 G^n(p, \theta) \text{ converges,}$$

$$(10) \quad \sum_n P_2 G^n(p, \theta) < \infty.$$

(We reverse the parametrization of $\gamma = \partial T$ if necessary.) Set $\theta_n = P_2 G^n(p, \theta)$. Using (9) we will show that there exist an L and an n' such that, for each $n \geq n'$,

$$(11) \quad \theta_{n+1} = \theta_n + d_n \theta_n^2$$

for some d_n such that $|d_n| \leq L$. Assume this for the time being.

We will show how this leads to a contradiction of (10).

A theorem of Weierstrass [5, p. 133] states that if the terms a_n of a given series $\sum a_n$ of complex terms possess for $n \geq m$ a representation of the form

$$a_{n+1}/a_n = 1 - \alpha/n + c_n$$

with $\sum |c_n| < \infty$, then $\sum a_n$ is convergent if and only if the real part of $\alpha > 1$. We apply this result with $a_n = \theta_n$, $m = n'$, $c_n = d_n \theta_n$, and $\alpha = 0$. From (10) we have $\sum \theta_n < \infty$. Since $|d_n| \leq L$ for $n \geq n'$ we have $\sum_{n=n'}^{\infty} |c_n| = \sum_{n=n'}^{\infty} |d_n \theta_n| \leq \sum_{n=n'}^{\infty} L \theta_n < \infty$. From (11) we have

$$\theta_{n+1}/\theta_n = 1 - 0/n + c_n.$$

Hence Weierstrass' theorem implies that $\sum \theta_n$ diverges. This contradicts (10).

It remains to establish (11). Consider two values of the arclength, $s < t$. Let α and β be, respectively, the small positive angles made by $\dot{\gamma}(s)$ and $\dot{\gamma}(t)$ with $\gamma(t) - \gamma(s)$. Expanding γ in a Taylor series about s we obtain

$$\begin{aligned}\sin \alpha &= \frac{[\dot{\gamma}(s), \gamma(t) - \gamma(s)]}{\|\gamma(t) - \gamma(s)\|} \\ &= \frac{[\dot{\gamma}(s), \dot{\gamma}(s)(t-s) + \frac{1}{2}\ddot{\gamma}(s)(t-s)^2 + \frac{1}{6}\ddot{\gamma}(\xi)(t-s)^3]}{\|\dot{\gamma}(s)(t-s) + \frac{1}{2}\ddot{\gamma}(\eta)(t-s)^2\|} \\ &= \frac{1}{2}K(s)(t-s) + o(t-s)^2\end{aligned}$$

where $s \leq \xi, \eta \leq t$ and we have used $\ddot{\gamma}(s) = K(s)n(s)$ and $\ddot{\gamma}(\xi) = K(\xi)n(\xi) - (K(\xi))^2\dot{\gamma}(\xi)$. Similarly

$$\sin \beta = \frac{1}{2}K(t)(t-s) + o(t-s)^2.$$

Hence by expanding arcsin in a Taylor series we have

$$(12) \quad \alpha = \frac{1}{2}K(s)(t-s) + o(t-s)^2, \quad \beta = \frac{1}{2}K(t)(t-s) + o(t-s)^2.$$

Thus

$$\begin{aligned}\beta - \alpha &= \frac{1}{2}(K(t) - K(s))(t-s) + o(t-s)^2 \\ &= \frac{1}{2}\dot{K}(\delta)(t-s)^2 + o(t-s)^2\end{aligned}$$

where $s \leq \delta \leq t$. Therefore

$$(13) \quad |\beta - \alpha| \leq L_1(t-s)^2$$

for some constants L_1 independent of s and t .

Since K is continuous periodic and always positive there must be by compactness a $\Delta > 0$ such that $K(r) \geq \Delta$ for all $r \in R$. In light of this it follows from (12) and (13) that there exist an $\epsilon > 0$ and an L such that

$$(14) \quad |\beta - \alpha| \leq L\alpha^2 \quad \text{for all } s, t \text{ such that } |t-s| \leq \epsilon.$$

Equation (11) now follows from (9) and (14). Q.E.D.

REMARK. The proof of Theorem 3 shows that $\lim_{n \rightarrow \infty} P_1^n G^n(p, \theta) = q$ is impossible if $K > 0$ at q and γ''' is bounded in a neighborhood of q .

We will use m to represent each of the following measures: Lebesgue measure on $\text{int } T \subset \mathbb{R}^2$ and on $(0, \pi) \subset \mathbb{R}^1$, arclength measure on the unit circle S and on ∂T , and the product measures $m \times m$ on $\text{int } T \times S$ and on $\partial T \times (0, \pi)$.

THEOREM 4. *On any billiard table T , for almost all initial conditions $(p, v) \in \text{int } T \times S$, $F_t(p, v)$ is well defined for all $t \geq 0$.*

PROOF. Assume $m(Q) > 0$ where $Q = \{(p, v) \in \text{int } T \times S \mid F_t(p, v) \text{ is not}$

well defined for some $t \geq 0$. Birkhoff shows in [3] that the measure μ on $\partial T \times (0, \pi)$ given by

$$(15) \quad \mu(A) = \int_A \sin \theta \, dm(p) \, dm(\theta)$$

(A an m -measurable subset of $\partial T \times (0, \pi)$) is preserved by G . The proof of this fact involves calculating the Jacobian of G which is found to be $\sin \theta / \sin \theta_1$ where $\theta_1 = P_2 G(p, \theta)$. From this it also can be seen that the rank of $H: \text{int } T \times S \rightarrow \partial T \times (0, \pi)$ is always 2 where $H(p, v) = (q, \theta)$ as shown in Figure 1.

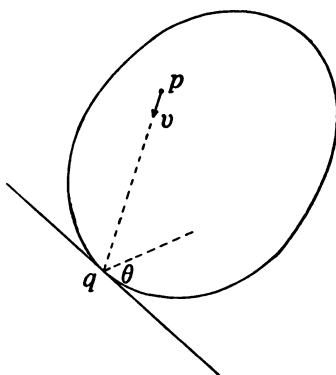


FIGURE 1

Hence H is locally like a projection of \mathbf{R}^3 onto \mathbf{R}^2 . It follows that $m(R) > 0$ where $R = H(Q) = \{(q, \theta) \in \partial T \times (0, \pi) \mid P_1 G^n(q, \theta) \text{ converges as } n \rightarrow \infty\}$. The second equality comes from Corollary 2. It follows from (15) that $\mu(R) > 0$. Let

$$U_\varepsilon = \{(q, \theta) \in \partial T \times (0, \pi) \mid \theta < \varepsilon \text{ or } \pi - \theta < \varepsilon\}.$$

Clearly $\mu(U_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Pick ε so that $\mu(U_\varepsilon) < \frac{1}{2}\mu(R)$. By Corollary 2 and Theorem 1, for each $(p, \theta) \in R$, $P_2 G^n(p, \theta) \rightarrow 0$ or π . Hence, for every $(p, \theta) \in R$ there is an n such that $G^{n'}(p, \theta) \in U_\varepsilon$ for all $n' \geq n$. Therefore

$$R \subset \bigcup_{n=1}^{\infty} \bigcap_{n' \geq n} G^{-n'}(U_\varepsilon)$$

and the union is an increasing one. Hence we can find an n such that

$$\mu\left(\bigcap_{n' \geq n} G^{-n'}(U_\varepsilon)\right) > \frac{1}{2}\mu(R).$$

Thus, $\mu(G^{-n}(U_\varepsilon)) > \frac{1}{2}\mu(R)$. But G preserves μ , so $\mu(G^{-n}(U_\varepsilon)) = \mu(U_\varepsilon)$

$> \frac{1}{2}\mu(R)$, a contradiction. Therefore $mQ = 0$. Q.E.D.

We leave the following trick shot to the reader. Given a table T with F_i not well defined and a point q outside the table, start the ball in the interior of T and hit the point q .

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